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# Modular Version of The Total Vertex Irregularity Strength for The Generalized Petersen Graph

Dina Khairani Nasution[1], Susilawati[2]

*[1,2]Department of Mathematics, Faculty of Mathematics and Science, Riau University*
*Campus Bina Widya Km. 12.5 Simpang Baru, Pekanbaru, 28293, Indonesia*

*Correspondense : Susilawati, Email:* susilawati.math@lecturer.unri.ac.id

### Abstrak

**Misalkan $G = (V, E)$ adalah suatu graf. Pelabelan graf merupakan suatu fungsi yang memetakan himpunan titik dan/sisi $G$ ke dalam himpunan bilangan bulat positif. Pelabelan total modular dikatakan pelabelan-$k$ total tak teratur titik modular dari $G$ jika untuk setiap dua titik berbeda $x$ dan $y$ di $G$ memiliki bobot modular yang berbeda, dan merupakan anggota himpunan bilangan bulat $\mathbb{Z}_n$. Nilai minimum $k$ sehingga graf $G$ memiliki pelabelan-$k$ total tak teratur titik modular disebut sebagai nilai total ketakteraturan titik modular, dinotasikan dengan $mtvs(G)$. Pada penelitian ini, dipelajari tentang nilai total ketakteraturan titik modular pada graf Petersen diperumum $(GP_{n,2})$. Hasil yang diperoleh menunjukkan nilai eksak $mtvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$.**

**Kata Kunci**: Graf Petersen diperumum, nilai total ketakteraturan titik modular, pelabelan total modular, pelabelan total tak teratur titik

### Abstract

**Let $G = (V, E)$ be a graph. A labeling graph is a maps function of the set of vertices and/or edges of $G$, to the set of positive integers. A total modular labeling is said to be a $k$-modular total irregular labeling of the vertices of $G$, if for every two distinct vertices $x$ and $y$ in $G$, the modular weights are different, and belong to the set of integers $\mathbb{Z}_n$. The minimum $k$ such that the graph $G$ has a $k$- modular total irregular labeling is called the modular total vertex irregularity strength and denoted by $mtvs(G)$. In this paper, we study about the modular total vertex irregularity strength for the generalized Petersen graph $(GP_{n,2})$. The result show that the exact value is $mtvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$.**

**Keywords**: Generalized Petersen graph, total vertex irregularity strength, total vertex irregular labeling, modular total vertex irregular labeling

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## Introduction

A graph labeling is a function that maps the elements of a graph to a set of positive integers. A graph labeling is called a vertex labeling if the domain of the mapping is the vertex set, called an edge labeling if the domain of the mapping is the edge set, and called a total labeling if the domain of the mapping is the set of both vertices and edges. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. In 1988, Chartrand et al. [1] introduced an irregular labeling, where for every two distinct vertices $x$ and $y$ in $V(G)$, the vertex weights are different.

For a graph $G$, in [2] introduced a total labeling $\varphi : V(G) \cup E(G) \to [1, k]$, such that each vertex has a distinct weight, where the total weight of a vertex $x$ is defined as $wt(x) = \varphi(x) + \sum_{xy \in E(G)} \varphi(xy)$. The minimum $k$ such that $G$ has a $k$-total vertex irregular labeling is called the *total vertex irregularity* *strength of* $G$. Some studies on this total vertex irregular labeling include by Baca et al. [3] and Ahmad et al. [4], who have determined the total irregularity strength for the generalized Petersen graph. In [5], Indriati et al. determined the total irregularity strength for double star graphs and several connected graphs. Other research on total vertex irregular labeling can be found in [6,7,8,17,18,19,21].

Furthermore, Baca et al. [9] and Gohar et al. [10] introduced the modular version of the irregular labeling and irregular total labeling. The modular version of irregular labeling involves applying a mapping function $\lambda : V(G) \to \mathbb{Z}_n$ defined by

$$\lambda(x) = wt(x) = \sum_{xy \in E(G)} f(xy) \ (mod\ n). \tag{1}$$

The label $\lambda(x)$ is the modular weight of the vertex $x$, $f$ is the edge labeling, and $\mathbb{Z}_n$ is the group of integers modulo $n$. Some studies on this irregular modular labeling include double star graphs, friendship graphs, wheels, fireworks, some flower graphs, and corona product graphs [11,12,13,14,15]. Additionally, Koam et al. [16] have determined the value of the modular edge irregularity for several classes of graphs.

The study about modular version of irregular total vertex labeling [10], is introduced, which defines the modular weight of the total vertex irregular labeling. For a graph $G$ with order $n$, the modular weight of a vertex $x$ in the labeling $f$ defined by

$$wt(x) = f(x) + \sum_{xy \in E(G)} f(xy) \ (mod\ n). \tag{2}$$

A total labeling $f: V(G) \cup E(G) \to [1, k]$ is said to be a modular total vertex irregular labeling of $G$ if for every two distinct vertices $x$ and $y$ in $V(G)$, it satisfies $wt(x) \neq wt(y)$. The modular total vertex irregularity strength of the graph $G$, denoted by $mtvs(G)$, is the minimum $k$ such that $G$ has a $k$-modular total vertex irregular labeling.

In this paper, we determine the modular total vertex irregularity strength for the generalized Petersen graph with $m = 2$, denoted by $(GP_{n,2})$, where $n \geq 5$, by determining the upper and lower bounds of $mtvs(GP_{n,2})$.

## Research Methods

The method used in this paper is a literature study by reviewing related research topics on modular total vertex irregular labeling. The steps in completing this research involve determining the lower and upper bounds of the modular total vertex irregularity strength for the generalized Petersen graph $(GP_{n,2})$, where $2 \leq 2m < n$, and then determining the exact value of $mtvs(GP_{n,2})$ through the previously obtained lower and upper bounds.

**Definition 1** [16] Let $G = (V, E)$ be a graph with no components of order lest than 2. A $k$-total labeling is said to be a modular total vertex irregular labeling of $G$ if there exists a bijective weight function $\lambda : V(G) \to \mathbb{Z}_n$, which is defined by

$$\lambda(u) = \psi(u) + \sum_{uv \in (G)} \psi(uv) \quad (mod\ n), \tag{3}$$

where $\lambda(u)$ is the modular weight of the vertex $u$, $\psi(u)$ is the label of the vertex $u$, and $\sum \psi(uv)$ is the sum of the labels of the edges that incident to the vertex $u$.

The minimum $k$ such that $G$ can be labeled with a $k$- modular total vertex irregular labeling is called the *modular total vertex irregularity strength*. Denoted by $mtvs(G)$. Furthermore, [14] provides a linear bound for $mtvs(G)$. Let $G$ be a graph with no components of order $\leq 2$, then

$$tvs(G) \leq mtvs(G). \tag{4}$$

**Theorem 2.** Let $G$ be a graph with $tvs(G) = k$. If the total vertex weights are corresponding to the $k$-total irregular labeling from a set of consecutive integers, then $tvs(G) = mtvs(G) = k$.

**Example 3.** Let $P_3$ be a path graph with a set of vertices $V(P_3) = \{v_1, v_2, v_3\}$ and a set of edges $E(P_3) = \{v_1v_2, v_2v_3\}$ as shown in Figure 1. Then, a labeling $f$ is defined on the $P_3$ such that
$f\{v_1\} = 1, f(v_2) = 1$, and $f(v_3) = 1$, then $f\{v_1v_2\} = 1$, and $f\{v_2v_3\} = 2$.
Furthermore, by using equation 3, we obtained that the modular weight of the vertex $v_1 = 2$, the modular weight of the vertex $v_2 = 1$, and the modular weight of the vertex $v_3 = 0$.

Based on the labelling $f$, each vertex in $P_3$ has a distinct modular vertex weight. Therefore, the labeling $f$ is a modular total vertex irregular labeling on $P_3$. Furthermore, the modular total vertex irregularity strength of $P_3$ is obtained to be $mtvs(P_3) = 2$.

The modular total vertex irregular labeling of the path graph $(P_3)$ can be illustrated in Figure 1 below.

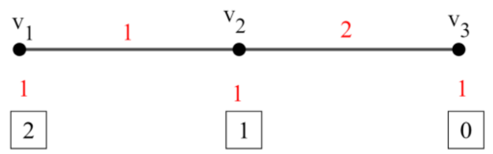


**Figure 1.** The modular total vertex irregular labeling of the $P_3$, with $mtvs(P_3) = 2$

## Result and Discussion

In this section we give the exact value of the modular total vertex irregularity for the generalized Petersen graph $(GP_{n,2})$, where $n \geq 5$.

**Definition 4.** The Petersen graph is a cubic graph of degree three with ten vertices and fifteen edges.

**Definition 5.** The generalized Petersen graph is a 3-regular graph as an extension of the Petersen graph, denoted by $GP_{n,m}$, where $n$ represents the number of vertices and $m$ represents the distance of the edge mapping, with $2 \leq 2m < n$.

The generalized Petersen graph has a set of vertices and edges:

$$V(GP_{n,m}) = \{v_i, v_i' \mid 1 \leq i \leq n\}$$

$$E(GP_{n,m}) = \{v_iv_{(i+1)}, v_i'v_{(i+m)}', v_iv_i' | 1 \leq i \leq n\},$$

where the indices $(i+1)$ and $(i+m)$ are taken modulo $n$. The generalized Petersen graph has two types of vertices: inner vertices, denoted by $v_i'$, and outer vertices, denoted by $v_i$. It also has three types of edges: outer edges, which connect vertices $v_i$ and $v_{(i+1)}$, inner edges, which connect vertices $v'_i$ and $v'_{(i+m)}$, and spokes, which connect vertices $v_i$ and $v_i'$ .

In this paper, we determine the modular total vertex irregularity strength for the generalized Petersen graph with $m = 2$, denoted by $(GP_{n,2})$. The generalized Petersen graph $(GP_{n,2})$ has a set of vertices and edges:

$$V(GP_{n,2}) = \{v_i, v_i' \mid 1 \leq i \leq n\}$$

$$E(GP_{n,2}) = \{v_i v_{(i+1)}, v_i' v_{(i+2)}', v_i v_i' | 1 \leq i \leq n\}.$$

where the indices $(i+1)$ and $(i+2)$ are taken modulo $n$. In general, the structure of the generalized Petersen graph with $m = 2$ can be seen in the following figure.

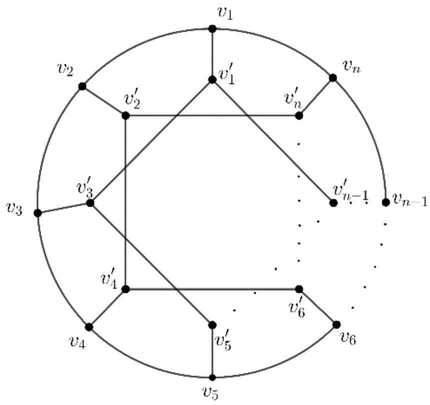

**Figure 2.** Generalized Petersen graph $GP_{n,2}$

**Modular Total Vertex Irregularity Strength Of Generalized Petersen Graph**

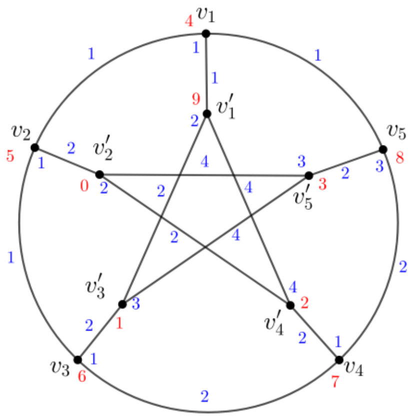

**Figure 3.** Modular total vertex irregularity strength of generalized Petersen graph $mtvs(GP_{5,2}) = 4$

**Theorem 6.** Let $G = (GP_{n,2})$ be the generalized Petersen graph with $n \geq 5$. The modular total vertex irregularity strength of the generalized Petersen graph $GP_{n,2}$, is $mtvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$.

**Proof.** To prove the exact value this theorem, we show that the lower bound of $mtvs(GP_{n,2}) \geq \left\lceil \frac{2n+3}{4} \right\rceil$ and the upper bound of $mtvs(GP_{n,2}) \leq \left\lceil \frac{2n+3}{4} \right\rceil$. Let $V$and $E$ be the vertex and edge of the generalized Petersen graph, repectively:

$$V(GP_{n,2}) = \{v_i, v_i' \mid 1 \leq i \leq n\}$$

$$E(GP_{n,2}) = \{v_i v_{(i+1)}, v_i' v_{(i+2)}', v_i v_i' | 1 \le i \le n\}.$$

where the indices $(i+1)$ and $(i+2)$ are taken modulo $n$.

From [10], it has has shown that $tvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$. Then by equation (4), we obtained that

$$mtvs(GP_{n,2}) \ge \left\lceil \frac{2n+3}{4} \right\rceil. \quad (5)$$

Now, we proved that $mtvs(GP_{n,2}) \ge \left\lceil \frac{2n+3}{4} \right\rceil$. Let $k = \left\lceil \frac{2n+3}{4} \right\rceil$, then we showed that $mtvs(GP_{n,2}) \le k$, by define $f$ as a labeling function from the set of vertices and edges in $GP_{n,2}$ with $n \ge 5$. To construct the labelling function of $f$, there is two cases below:

Case 1. *For $n$ is even*

$$f(v_i) = \begin{cases} 1, & if \ \ 1 \le i \le 4 \\ 2, & if \ \ 5 \le i \le n-1 \\ \left\lfloor \dfrac{i-k}{2} \right\rfloor + 2, & if \ \ i = n \end{cases}$$

$$f(v_i') \begin{cases} 2\left(\dfrac{i+1}{4}\right) + 1, & if \ \ i \equiv 3 \ (mod\ 4), i \ne n \\ \dfrac{i+2}{2}, & if \ \ i = n \\ 2 + 2\left\lfloor \dfrac{i}{4} \right\rfloor, & otherwise \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 1, & if \ \ 1 \le i \le 2, i = n \\ \left\lfloor \dfrac{i-3}{4} \right\rfloor + 2, & if \ \ 2 < i \le n-1 \end{cases}$$

$$f(v_i' v_{i+2}') = \begin{cases} k-1, & if \ \ 1 \le i \le 2 \\ k, & if \ \ 2 < i \le n. \end{cases}$$

Then, for $6 \le n \le 10$ we define

$$f(v_i v_i') = \begin{cases} i, & if \ \ 1 \le i \le 2 \\ \left\lfloor \dfrac{i-2}{4} \right\rfloor + 2, & if \ \ 2 < i \le n-2 \\ \left\lfloor \dfrac{i}{2} \right\rfloor, & if \ \ n-1 < i \le n, \end{cases}$$

and for $n \ge 11$

$$f(v_i v_i') = \begin{cases} \dfrac{i-1}{2}, & if \ \ i \equiv 1 \ (mod\ 4), i \ne 1, i \ne n \\ \left\lceil \dfrac{i-1}{4} \right\rceil + 1, & if \ \ 1 \le i \le 8 \\ \dfrac{i}{2}, & if \ \ i = n \\ 5 + 2\left\lfloor \dfrac{i-9}{4} \right\rfloor, & otherwise. \end{cases}$$

Case 2. *For $n$ is odd*

$$f(v_i) = \begin{cases} \left\lceil \frac{i}{4} \right\rceil, & if \;\; 1 \leq i \leq n-1 \\ \left\lfloor \frac{i-k}{2} \right\rfloor + 3, & if \;\; i = n \end{cases}$$

$$f(v_i') = \begin{cases} 2, & if \;\; i = 1 \\ i, & if \;\; 2 \leq i \leq 4 \\ 3\left(\frac{i-1}{4}\right), & if \;\; i \equiv 1 \,(mod\; 4), i \neq 1, i \neq n \\ k-1, & if \;\; i = n \\ f(v_{i-1}') + 1, & otherwise \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 1, & if \;\; 1 \leq i \leq 2, i = n \\ \left\lfloor \frac{i-3}{4} \right\rfloor + 2, & if \;\; 2 < i \leq n-1 \end{cases}$$

$$f(v_i' v_{i+2}') = \begin{cases} k-2, & if \;\; 1 \leq i \leq 2 \\ k, & if \;\; 2 < i \leq n \end{cases}$$

$$f(v_i v_i') = \begin{cases} \left\lceil \frac{i-1}{4} \right\rceil + 1, & if \;\; 1 \leq i \leq n-1 \\ i-k+1, & if \;\; i = n. \end{cases}$$

Based on the labelling function $f$ that has been defined, it is obtained for all vertices $v, v_i' \in V$, $wt(v) = f(v) + \sum f(vv') \;\; (mod\; n)$, the modular weights are different. That is $wt(v_1) < wt(v_2) < wt(v_3) < \cdots wt(v_n) < wt(v_1') < wt(v_2') < wt(v_3') < \cdots < wt(v_n')$. Then, based on definition, the labeling $f$ is a modular total vertex irregular labeling on $GP_{n,2}$ and it is obtained that the largest label used is $k = \left\lceil \frac{2n+3}{4} \right\rceil$, such that

$$mtvs(GP_{n,2}) \leq \left\lceil \frac{2n+3}{4} \right\rceil, \tag{6}$$

Therefore, according to (5) and (6) we conclude that $mtvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$. □

Figure 4 shows the modular total vertex irregular labeling on the generalized Petersen graph with $n = 6, 7,$ and $8$.

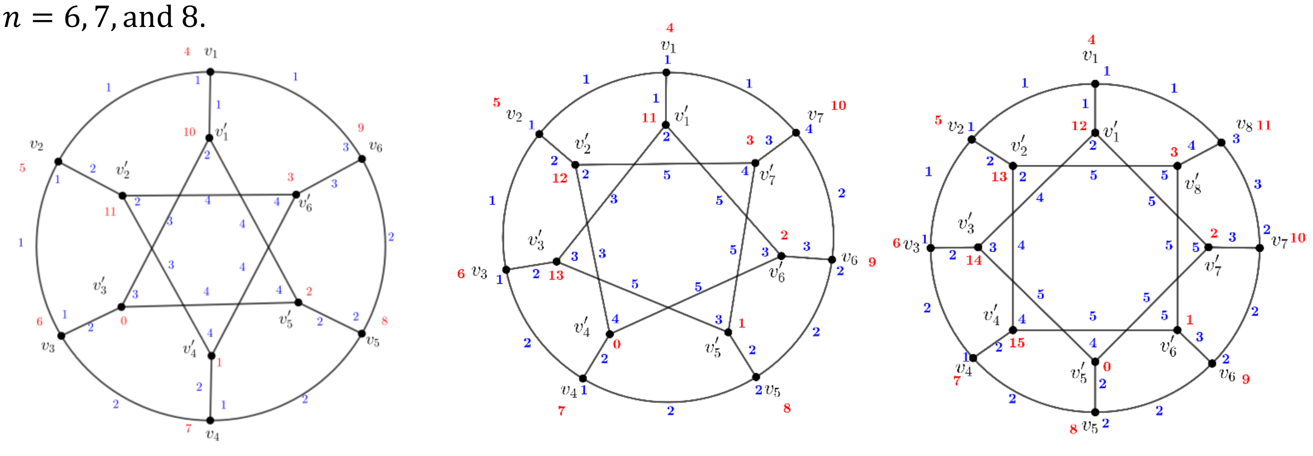

**Figure 4.** Modular total vertex irregularity strength of generalized Petersen graph $GP_{n,2}$ for $n = 6, 7,$ and $8$.

## Conclusion

A modular total vertex irregular labeling is a function that maps the set of vertices and edges of a graph to the set of positive integers, such that each vertex has a distinct modular weight. The generalized Petersen graph is a 3-regular graph as an extension of the Petersen graph, denoted by $(GP_{n,m})$, where $2 \leq 2m < 2.$ In this paper, we obtained the exact value of the modular total vertex irregularity strength of $GP_{n,2}$ is $mtvs(GP_{n,2}) = \left\lceil \frac{2n+3}{4} \right\rceil$.

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