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Generalized Gaussian Fibonacci Numbers and its Determinantal Identities

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Abstract

In this paper, we present the determinantal identities of generalized Gaussian Fibonacci numbers. The generalized Gaussian Fibonacci sequence $Gf_{n+2}(p,q;a,b)$ is defined by the recurrence relation $Gf_{n+2} = pGf_{n+1} + qGf_n$, with $Gf_0 = a$ and $Gf_1 = b$. This was introduced by S. Pethe and A. F. Horadam. Also, we present its determinantal identities with classical numbers like gaussian Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, jacobsthal-Lucas, Bronze, Nickel and Mersenne numbers.

Keywords: Gaussian Fibonacci number, Gaussian Lucas number, Gaussian Pell number, Gaussian Pell-Lucas number, Gaussian Jacobsthal number, Gaussian jacobsthal-Lucas number and determinant.

Introduction

Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinants.

Complex Fibonacci numbers [1], often known as Gaussian Fibonacci numbers, were first presented by Horadam [2] in 1963. In 1965, Jordan [3] studied two sequences of complex numbers and came up with certain traits of regular Fibonacci sequences.

Gaussian Fibonacci numbers [3], are defined by for $n \geq 2$,

$$GF_n = GF_{n-1} + GF_{n-2} \tag{1}$$

With $GF_0 = i, \;\; GF_1 = 1$. One can see that

$$GF_n = F_n + iF_{n-1} \tag{2}$$

Where F_n is the nth Fibonacci number.

Gaussian Lucas numbers [3], are defined by for $n \geq 2$,

$$GL_n = GL_{n-1} + GL_{n-2} \tag{3}$$

With $GL_0 = 2 - i, \;\; GL_1 = 1 + 2i$. One can see that

$$GL_n = L_n + iL_{n-1} \tag{4}$$

Where L_n is the nth Lucas number.

The Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell–Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal–Lucas, Gaussian Bronze, Gaussian Nickel, and Gaussian Mersenne numbers are examples of new complex number sequences that are created by combining recursively defined numerical sequences with Gaussian type integers. In this paper, we present generalized Gaussian Fibonacci numbers and its determinantal identities. We also establish results in terms of gaussian Fibonacci numbers, gaussian Lucas numbers, gaussian Pell numbers, gaussian Pell-Lucas numbers, gaussian Jacobsthal numbers, gaussian jacobsthal-Lucas numbers.

Generalized Gaussian Fibonacci Sequence

Definition 1. The generalized Gaussian Fibonacci sequence [4, 5, 6], is defined by the recurrence relation

$$Gf_{n+2} = pGf_{n+1} + qGf_n \tag{5}$$

with $Gf_0 = a$ and $Gf_1 = b$, where $a \; and \; b$ are initial values.

First few generalized Gaussian Fibonacci numbers are

$Gf_2 = pb + qa$

$Gf_3 = p^2b + pqa + qb$

$Gf_4 = p^3b + p^2qa + 2pqb + q^2a$

$Gf_5 = p^4b + p^3qa + 3p^2bq + 2p^2qa + pqb$

…

If $a = i, b = 1, p = 1, q = 1$, then Gaussian Fibonacci numbers

$$GF_n = GF_n + GF_{n-1}$$

If $a = i, b = 1, p = 2, q = 1$, then Gaussian Pell numbers [7]

$$GP_n = 2GP_n + GP_{n-1}$$

If $a = i/2, b = 1, p = 1, q = 2$, then Gaussian Jacobsthal numbers [8]

$$GJ_n = GJ_n + 2GJ_{n-1}$$

If $a = i, b = 1, p = 3, q = 1$, then Gaussian Bronze numbers [9], [10]

$$GB_n = 3GB_n + GB_{n-1}$$

If $a = i/3, b = 1, p = 1, q = 3$, then Gaussian Nickel numbers [11]

$$GN_n = GN_{n-1} + 3GN_{n-2}$$

If $a = -\frac{i}{2}, b = 1, p = 3, q = -2$, then Gaussian Mersenne numbers [12]

$$GM_n = 3GM_n - 2GM_{n-1}$$

If $a = 2 - i, b = 1 + 2i, p = 1, q = 1$, then Gaussian Lucas numbers [13]

$$GL_n = GL_n + GL_{n-1}$$

If $a = 2 - 2i, b = 2 + 2i, p = 2, q = 1$, then Gaussian Pell-Lucas numbers [14]

$$GQ_n = 2GQ_n + GQ_{n-1}$$

If $a = 2 - i/2, b = 1 + 2i, p = 1, q = 2$, then Gaussian Jacobsthal-Lucas numbers [8]

$$Gj_n = Gj_n + 2Gj_{n-1}$$

Determinantal Identities

Determinants of matrices with generalized fibonaci entries are present on research [15]. In this discussion, some results are given about the determinant of matrices related to the generalized Gaussian Fibonacci number that has been defined previously.

Theorem 2. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} 0 & qGf_{n+1}p^2Gf_{n+2}^2 & qGf_{n+1}Gf_{n+3}^2 \\ q^2Gf_{n+1}^2pGf_{n+2} & 0 & pGf_{n+2}Gf_{n+3}^2 \\ q^2Gf_{n+1}^2Gf_{n+3} & p^2Gf_{n+2}^2Gf_{n+3} & 0 \end{vmatrix} = 2q^3Gf_{n+1}^3p^3Gf_{n+2}^3Gf_{n+3}^3$$

Proof. Let $\Delta = \begin{vmatrix} 0 & qGf_{n+1}p^2Gf_{n+2}^2 & qGf_{n+1}Gf_{n+3}^2 \\ q^2Gf_{n+1}^2pGf_{n+2} & 0 & pGf_{n+2}Gf_{n+3}^2 \\ q^2Gf_{n+1}^2Gf_{n+3} & p^2Gf_{n+2}^2Gf_{n+3} & 0 \end{vmatrix}$

Assume $qGf_{n+1} = m$ and $pGf_{n+2} = n$, then by the definition of generalized Gaussian Fibonacci sequence $Gf_{n+3} = m + n$, now [1]

$$\Delta = \begin{vmatrix} 0 & mn^2 & m(m+n)^2 \\ m^2n & 0 & n(m+n)^2 \\ m^2(m+n) & n^2(m+n) & 0 \end{vmatrix}$$

Taking m, n and $(m+n)$ $as\ common\ from\ R_1, R_2, R_3\ respectively$

$$\Delta = mn(m+n) \begin{vmatrix} 0 & n^2 & (m+n)^2 \\ m^2 & 0 & (m+n)^2 \\ m^2 & n^2 & 0 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$,

$$\Delta = mn(m+n) \begin{vmatrix} 0 & n^2 & (m+n)^2 \\ 0 & -n^2 & (m+n)^2 \\ m^2 & n^2 & 0 \end{vmatrix}$$

Expanding by R_3,

$$\Delta = m^3 n(m+n)\begin{vmatrix} n^2 & (m+n)^2 \\ -n^2 & (m+n)^2 \end{vmatrix}$$

$$\Delta = m^3 n(m+n)[n^2(m+n)^2 + n^2(m+n)^2]$$

$$\Delta = m^3 n^3 (m+n)^3$$

Put $qGf_{n+1} = m, pGf_{n+2} = n$ and $Gf_{n+3} = m+n$, we get

$\Delta = 2q^3 Gf_{n+1}^3 p^3 Gf_{n+2}^3 Gf_{n+3}^3$

This completes the Proof.

Corollary 3. If $a = i, b = 1, p = 1, q = 1$, then Gaussian Fibonacci numbers

$$\begin{vmatrix} 0 & GF_{n+1}GF_{n+2}^2 & GF_{n+1}GF_{n+3}^2 \\ GF_{n+1}^2 GF_{n+2} & 0 & GF_{n+2}GF_{n+3}^2 \\ GF_{n+1}^2 GF_{n+3} & GF_{n+2}^2 GF_{n+3} & 0 \end{vmatrix} = 2GF_{n+1}^3 GF_{n+2}^3 GF_{n+3}^3$$

Corollary 4. If $a = i, b = 1, p = 2, q = 1$, then Gaussian Pell numbers

$$\begin{vmatrix} 0 & GP_{n+1}4GP_{n+2}^2 & GP_{n+1}GP_{n+3}^2 \\ GP_{n+1}^2 2GP_{n+2} & 0 & 2GP_{n+2}GP_{n+3}^2 \\ GP_{n+1}^2 GP_{n+3} & 4GP_{n+2}^2 GP_{n+3} & 0 \end{vmatrix} = 16GP_{n+1}^3 GP_{n+2}^3 GP_{n+3}^3$$

Corollary 5. If $a = i/2, b = 1, p = 1, q = 2$, then Gaussian Jacobsthal numbers

$$\begin{vmatrix} 0 & 2GJ_{n+1}GJ_{n+2}^2 & 2GJ_{n+1}GJ_{n+3}^2 \\ 4GJ_{n+1}^2 GJ_{n+2} & 0 & GJ_{n+2}GJ_{n+3}^2 \\ 4GJ_{n+1}^2 GJ_{n+3} & GJ_{n+2}^2 GJ_{n+3} & 0 \end{vmatrix} = 16GJ_{n+1}^3 GJ_{n+2}^3 GJ_{n+3}^3$$

Corollary 6. If $a = i, b = 1, p = 3, q = 1$, then Gaussian Bronze numbers

$$\begin{vmatrix} 0 & GB_{n+1}9GB_{n+2}^2 & GB_{n+1}GB_{n+3}^2 \\ GB_{n+1}^2 3GB_{n+2} & 0 & GB_{n+2}GB_{n+3}^2 \\ GB_{n+1}^2 GB_{n+3} & 9GB_{n+2}^2 GB_{n+3} & 0 \end{vmatrix} = 54GB_{n+1}^3 GB_{n+2}^3 GB_{n+3}^3$$

Corollary 7. If $a = i/3, b = 1, p = 1, q = 3$, then Gaussian Nickel numbers

$$\begin{vmatrix} 0 & 3GN_{n+1}GN_{n+2}^2 & 3GN_{n+1}GN_{n+3}^2 \\ 9GN_{n+1}^2 GN_{n+2} & 0 & GN_{n+2}GN_{n+3}^2 \\ 9GN_{n+1}^2 GN_{n+3} & Gf_{n+2}^2 Gf_{n+3} & 0 \end{vmatrix} = 54GN_{n+1}^3 GN_{n+2}^3 GN_{n+3}^3$$

Corollary 8. If $a = -\frac{i}{2}, b = 1, p = 3, q = -2$, then Gaussian Mersenne numbers

$$\begin{vmatrix} 0 & -2GM_{n+1}9GM_{n+2}^2 & -2GM_{n+1}GM_{n+3}^2 \\ 4GM_{n+1}^2 3GM_{n+2} & 0 & 3GM_{n+2}GM_{n+3}^2 \\ 4GM_{n+1}^2 GM_{n+3} & 9GM_{n+2}^2 GM_{n+3} & 0 \end{vmatrix} = -432GM_{n+1}^3 GM_{n+2}^3 GM_{n+3}^3$$

Corollary 9. If $a = 2 - i, b = 1 + 2i, p = 1, q = 1$, then Gaussian Lucas numbers

$$\begin{vmatrix} 0 & GL_{n+1}GL_{n+2}^2 & GL_{n+1}GL_{n+3}^2 \\ GL_{n+1}^2 GL_{n+2} & 0 & GL_{n+2}GL_{n+3}^2 \\ GL_{n+1}^2 GL_{n+3} & GL_{n+2}^2 GL_{n+3} & 0 \end{vmatrix} = 2GL_{n+1}^3 GL_{n+2}^3 GL_{n+3}^3$$

Corollary 10. If $a = 2 - 2i, b = 2 + 2i, p = 2, q = 1$, then Gaussian Pell-Lucas numbers

$$\begin{vmatrix} 0 & qGf_{n+1}p^2 Gf_{n+2}^2 & qGf_{n+1}Gf_{n+3}^2 \\ q^2 Gf_{n+1}^2 pGf_{n+2} & 0 & pGf_{n+2}Gf_{n+3}^2 \\ q^2 Gf_{n+1}^2 Gf_{n+3} & p^2 Gf_{n+2}^2 Gf_{n+3} & 0 \end{vmatrix} = 2q^3 Gf_{n+1}^3 p^3 Gf_{n+2}^3 Gf_{n+3}^3$$

Corollary 11. If $a = 2 - i/2, b = 1 + 2i, p = 1, q = 2$, then Gaussian Jacobsthal-Lucas numbers

$$\begin{vmatrix} 0 & 2Gj_{n+1}Gj_{n+2}^2 & 2Gj_{n+1}Gj_{n+3}^2 \\ 4Gj_{n+1}^2 Gj_{n+2} & 0 & Gj_{n+2}Gj_{n+3}^2 \\ 4Gj_{n+1}^2 Gj_{n+3} & Gj_{n+2}^2 Gj_{n+3} & 0 \end{vmatrix} = 16Gj_{n+1}^3 Gj_{n+2}^3 Gj_{n+3}^3$$

The proof of the theorems 12 to 18, are in line with the proof of Theorem 2.

Theorems 2, 12, 13, 14, and 15 use determinant-based identities to uncover profound and diverse structural characteristics of extended Gaussian Fibonacci numbers. They give a robust foundation for generalizing such results to other number sequences and algebraic systems.

Theorem 12. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} \{pGf_{n+2} + Gf_{n+3}\}^2 & q^2 Gf_{n+1}^2 & q^2 Gf_{n+1}^2 \\ p^2 Gf_{n+2}^2 & \{qGf_{n+1} + Gf_{n+3}\}^2 & p^2 Gf_{n+2}^2 \\ Gf_{n+3}^2 & Gf_{n+3}^2 & \{qGf_{n+1} + pGf_{n+2}\}^2 \end{vmatrix}$$
$$= 3pqGf_{n+1}Gf_{n+2}Gf_{n+3}\{qGf_{n+1} + pGf_{n+2} + Gf_{n+3}\}^3$$

Theorem 13. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} qGf_{n+1} + pGf_{n+2} & pGf_{n+2} + Gf_{n+3} & qGf_{n+1} + Gf_{n+3} \\ pGf_{n+2} + Gf_{n+3} & qGf_{n+1} + Gf_{n+3} & qGf_{n+1} + pGf_{n+2} \\ qGf_{n+1} + Gf_{n+3} & qGf_{n+1} + pGf_{n+2} & pGf_{n+2} + Gf_{n+3} \end{vmatrix}$$
$$= 6pqGf_{n+1}Gf_{n+2}Gf_{n+3} - 2[\{qGf_{n+1}\}^3 + \{pGf_{n+2}\}^3 + \{Gf_{n+3}\}^3]$$

Theorem 14. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} 2Gf_{n+3}+qGf_{n+1}+pGf_{n+2} & qGf_{n+1} & pGf_{n+2} \\ Gf_{n+3} & 2qGf_{n+1}+pGf_{n+2}+Gf_{n+3} & pGf_{n+2} \\ Gf_{n+3} & qGf_{n+1} & 2pGf_{n+2}+qGf_{n+1}+Gf_{n+3} \end{vmatrix} = 2\{qGf_{n+1}+pGf_{n+2}+Gf_{n+3}\}^3$$

Theorem 15. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} pGf_{n+2}+Gf_{n+3} & qGf_{n+1}+pGf_{n+2} & qGf_{n+1} \\ Gf_{n+3}+qGf_{n+1} & pGf_{n+2}+Gf_{n+3} & pGf_{n+2} \\ qGf_{n+1}+pGf_{n+2} & Gf_{n+3}+qGf_{n+1} & Gf_{n+3} \end{vmatrix} = q^3Gf_{n+1}^3+p^3Gf_{n+2}^3+Gf_{n+3}^3-3pqGf_{n+1}Gf_{n+2}Gf_{n+3}$$

Theorems 16, 17 and 18 arrive at the same elegant result, highlighting a deep consistency in the behavior of determinants constructed from these sequences. This reinforces the strong algebraic structure and recurrence symmetry inherent in generalized Gaussian Fibonacci numbers.

Theorem 16. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} p^2Gf_{n+2}^2+Gf_{n+3}^2 & pqGf_{n+1}Gf_{n+2} & qGf_{n+1}Gf_{n+3} \\ pqGf_{n+1}Gf_{n+2} & q^2Gf_{n+1}^2+Gf_{n+3}^2 & pGf_{n+2}Gf_{n+3} \\ qGf_{n+1} & pGf_{n+2}Gf_{n+3} & q^2Gf_{n+1}^2+p^2Gf_{n+2}^2 \end{vmatrix} = \{2pqGf_{n+1}Gf_{n+2}Gf_{n+3}\}^2$$

Theorem 17. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} -q^2Gf_{n+1}^2 & pqGf_{n+1}Gf_{n+2} & qGf_{n+1}Gf_{n+3} \\ pqGf_{n+1}Gf_{n+2} & -p^2Gf_{n+2}^2 & pGf_{n+2}Gf_{n+3} \\ qGf_{n+1}Gf_{n+3} & pGf_{n+2}Gf_{n+3} & -Gf_{n+3}^2 \end{vmatrix} = \{2pqGf_{n+1}Gf_{n+2}Gf_{n+3}\}^2$$

Theorem 18. If Gf_n is the generalized Gaussian Fibonacci numbers, then

$$\begin{vmatrix} q^2Gf_{n+1}^2 & pGf_{n+2}Gf_{n+3} & Gf_{n+3}^2+qGf_{n+1}Gf_{n+3} \\ q^2Gf_{n+1}^2+pqGf_{n+1}Gf_{n+2} & p^2Gf_{n+2}^2 & qGf_{n+1}Gf_{n+3} \\ pqGf_{n+1}Gf_{n+2} & p^2Gf_{n+2}^2+pGf_{n+2}Gf_{n+3} & Gf_{n+3}^2 \end{vmatrix} = \{2pqGf_{n+1}Gf_{n+2}Gf_{n+3}\}^2$$

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Conclusion

This paper developed determinant identities of generalized Gaussian Fibonacci numbers in generalized form. Also results derived in terms of classical Gaussian numbers like Fibonacci, Lucas, Pell, Pell-Lucas, Bronze, Nickel, and Mersenne numbers. We have obtained recursive results in all determinantal identities. These identities can be used to developed new identities for classical polynomials. These findings are also in line with previous studies on determinants of matrices using general Fibonacci-type sequences.

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