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On continuity properties of the improved conformable fractional derivatives

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Abstrak

Turunan fraksional *conformable* telah dikenalkan dengan menggunakan definisi limit yang mirip dengan turunan klasik. Selain mempunyai beberapa kelebihan dibandingkan dengan turunan fraksional lainnya seperti pemenuhan sifat-sifat seperti halnya turunan klasik dan kemudahan diselesaikan secara numerik, turunan fraksional *conformable* juga mempunyai kekurangan yaitu memberikan eror yang cukup besar dibandingkan dengan turunan fraksional Riemann-Liouville dan Caputo. Hasil modifikasi dari turunan fraksional *conformable* telah dikenalkan untuk menanggulangi kekurangan tersebut. Turunan fraksional *conformable* yang telah dimodifikasi dianggap sebagai hampiran terbaik untuk turunan fraksional Riemann-Liouville dan Caputo dalam hal perilaku fisis. Sifat-sifat kekontinuan fungsi yang berhubungan dengan turunan fraksional *conformable* dibahas dalam artikel ini. Hubungan antara turunan fraksional orde α dan kekontinuan suatu fungsi dibuktikan secara rinci. Selain itu, dalam artikel ini diberikan teorema yang mirip dengan teorema Rolle dan teorema nilai rata-rata untuk turunan fraksional *conformable* yang dimodifikasi.

Kata Kunci: kekontinuan, modifikasi, teorema nilai rata-rata, teorema Rolle, turunan fraksional conformable

Abstract

The conformable fractional derivative has been introduced to extend the familiar limit definition of the classical derivative. Despite having many advantages compared to other fractional derivatives such as satisfying nice properties as classical derivative and easy to solve numerically, it also has disadvantages as it gives large error compared to Riemann-Liouville and Caputo fractional derivatives. Modified types of conformable derivatives have been proposed to overcome the shortcoming. The improved conformal fractional derivatives are declared to be better approximations of Riemann-Liouville and Caputo derivatives in terms of physical behavior. In this paper, properties concerning continuity of the improved conformable fractional derivative are investigated. We prove the relation between α-differentiable and continuity of a function and corresponding interior extremum theorem. We also prove the properties close to Rolle's Theorem and Mean Value Theorem for the improved conformable fractional derivatives.

Keywords: conformable fractional derivative ; continuity; mean value theorem; modified, Rolle's theorem

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Introduction

Fractional calculus was introduced many centuries ago, but it is still growing splendidly up to now. It can be seen from numerous publications concerning fractional derivatives and integrals and its applications to mathematical modelling in many fields. Failla and Zingales in [1] stated that some studies have pointed out that fractional operators can successfully portrayed complex long-memory and multiscale phenomena in materials that can faintly captured by classical differential calculus. Acioli et al. in [2] used fractional derivatives to model the dispersion of pollutants in the planetary boundary layer. In epidemics, Shaikh et al. [3] used fractional derivative to model the outbreak of COVID-19 in India, while Moya et al. [4] to model tuberculosis with considering the relationship with HIV/AIDS and diabetes. Many other papers such as in [5]–[8] revealed the importance of using fractional derivatives to approach the models of the encountered problems.

Fractional derivatives are defined in various ways and mostly by fractional integrals. Some prominent definitions of fractional derivatives such as Riemann-Liouville and Caputo fractional derivatives have been investigated extensively for both its properties and applications (see for instance [9]–[13]). As it is defined using fractional integrals, Riemann-Liouville and Caputo fractional derivatives satisfy linear property as the classical derivative but not product and quotient rules. Those fractional derivatives also do not have corresponding Rolle's theorem and mean value theorem. Khalil et al in [14] introduced a new definition of fractional derivative called conformable derivative that employed familiar limit definition analogous to the classical derivative. The result of this definition revealed that conformable derivative satisfies product and quotient rules. It also has corresponding Rolle's theorem and mean value theorem. Hasanah et al. in [15] modified Fourier transform to handle fractional partial differential equations using conformal derivative. However, conformable fractional derivatives have disadvantages compared to Riemann-Liouville and Caputo fractional derivatives. Conformable derivative gives a different physical meaning as it offers large error compared to Riemann-Liouville and Caputo fractional derivatives.

In 2020, Gao and Chi [16] proposed the improvement on the definition of conformable fractional derivative to overcome the shortcoming. The authors introduced some improvements on the conformable derivative based on Riemann-Liouville and Caputo fractional derivatives. The result shows that the improved conformable derivative is a good approximation to the classical Riemann-Liouville or Caputo fractional derivative. In terms of numerical computing, it is considered as easy to compute as it is a local derivative. In this paper, the relationship between continuity and α differentiable is investigated. We also give theorems close to Rolle's theorem and mean value theorem involving the improved conformable fractional derivative.

Preliminaries

In this section we recall some basic definitions and properties of conformable derivative and the improved conformable fractional derivatives. Some fractional derivatives of elementary functions are also given and will be used to investigate properties regarding continuity of α-differentiable functions.

Definition 1. [14] Given a function $f: [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0,1)$.

Khalil et al. [14] showed that the conformable fractional derivative satisfies the continuity of α-differentiable functions. The authors also give corresponding Rolle's theorem and mean value theorem for the conformable fractional derivative.

Two types of fractional derivatives which have been extensively studied up to now are Riemann-Liouville and Caputo fractional derivatives. Different from the definition of conformable fractional derivative, Riemann-Liouville and Caputo fractional derivatives are defined using fractional integrals as follows.

Definition 2. For $\alpha \in [n-1, n)$, the α-derivative of f is defined as

(1) Riemann-Liouville

$$({}_{a}^{RL}D^{\alpha}f)(t)=\frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}\frac{f(x)}{(t-x)^{\alpha-n+1}}dx.$$

(2) Caputo

$$({}_{a}^{C}D^{\alpha}f)(t)=\frac{1}{\Gamma(\mathrm{n}-\alpha)}\int_{a}^{t}\frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx.$$

Both Riemann-Liouville and Caputo fractional integrals utilized fractional integrals, thus both have nonlocal behaviors including historical memory and future independence. However, both types of fractional derivatives do not have a corresponding Rolle's theorem and a mean value theorem.

Gao and Chi [16] introduced a kind of modified conformable derivative called the improved conformable derivative. In this definition, there are two types of modified definitions based on Riemann-Liouville and Caputo fractional derivatives. The improved Caputo-type conformable fractional derivative is given in Definition 3 and the improved Riemann-Liouville-type conformable fractional derivative is in Definition 4 as follows.

Definition 3. [16] Let $f:\mathbb{R}\to\mathbb{R}$ be a function. The improved Caputo-type conformable fractional derivative of f of order α is defined by

(1) For $0\le\alpha\le 1$,

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(t)=\lim_{\varepsilon\to 0}\left[(1-\alpha)\big(f(t)-f(a)\big)+\alpha\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}\right],$$

where $-\infty<a<t<+\infty$, a is a given number.

(2) For $n<\alpha\le n+1$, $n=1,2,\ldots$,

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(t)=\lim_{\varepsilon\to 0}\Bigg[(n+1-\alpha)\left(f^{(n)}(t)-f^{(n)}(a)\right)+(\alpha-n)\frac{f^{(n)}(t+\varepsilon(t-a)^{n+1-\alpha})-f^{(n)}(t)}{\varepsilon}\Bigg],$$

where $-\infty<a<t<+\infty$, a is a given number.

Definition 4. [16] Let $f:\mathbb{R}\to\mathbb{R}$ be a function. The improved Riemann-Liouville-type conformable fractional derivative is defined by

(1) For $0\le\alpha\le 1$,

$$\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(t)=\lim_{\varepsilon\to 0}\left[(1-\alpha)f(t)+\alpha\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}\right],$$

where $-\infty<a<t<+\infty$, a is a given number.

(2) For $n<\alpha\le n+1$, $n=1,2,\ldots$,

$$\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(t)=\lim_{\varepsilon\to 0}\left[(n+1-\alpha)f^{(n)}(t)+(\alpha-n)\frac{f^{(n)}(t+\varepsilon(t-a)^{n+1-\alpha})-f^{(n)}(t)}{\varepsilon}\right],$$

where $-\infty<a<t<+\infty$, a is a given number.

From the Definition 3 and Definition 4, if f is differentiable at t then we can deduce that for $\alpha\in(0,1]$,

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(t)=(1-\alpha)\big(f(t)-f(a)\big)+\alpha(t-a)^{1-\alpha}f'(t),$$

$$\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(t)=(1-\alpha)f(t)+\alpha(t-a)^{1-\alpha}f'(t).$$

While for $\alpha \in (n, n+1], n = 1,2, \dots$ and f is $(n+1)$-differentiable at t then

$$\left({}_a^C\tilde{T}_\alpha f\right)(t) = (n+1-\alpha)\left(f^{(n)}(t) - f^{(n)}(a)\right) + (\alpha - n)(t-a)^{n+1-\alpha} f^{(n+1)}(t).$$

Based on the definition, for $\alpha \in (0,1]$, the improved conformable fractional derivative satisfies the linear property as the classical derivative as stated in the following theorem.

Theorem 5. [16] Let $f, g: \mathbb{R} \to \mathbb{R}$ be functions and $0 \le \alpha \le 1$. The improved conformable fractional derivative satisfies the following.

(1) ${}_a^C\tilde{T}_\alpha(mf + ng) = m\,{}_a^C\tilde{T}_\alpha f + n\,{}_a^C\tilde{T}_\alpha g$

(2) ${}_a^{RL}\tilde{T}_\alpha(mf + ng) = m\,{}_a^{RL}\tilde{T}_\alpha f + n\,{}_a^{RL}\tilde{T}_\alpha g$.

In the following theorem we give the improved conformable fractional derivative of some elementary functions. This result will be used in the continuity properties which will be discussed in the next section.

Theorem 6. [16] Let $0 \le \alpha \le 1$. The improved conformable fractional derivatives of some elementary functions are given as follows.

(1) ${}_0^C\tilde{T}_\alpha(t^p) = {}_0^{RL}\tilde{T}_\alpha(t^p) = (1-\alpha)t^p + \alpha p t^{p-\alpha}$

(2) ${}_0^C\tilde{T}_\alpha(\lambda) = 0$, for any constant λ

(3) ${}_0^{RL}\tilde{T}_\alpha(\lambda) = (1-\alpha)\lambda$, for any constant λ

(4) ${}_0^C\tilde{T}_\alpha(e^t) = (1-\alpha)(e^t - 1) + \alpha t^{1-\alpha} e^t$

(5) ${}_0^{RL}\tilde{T}_\alpha(e^t) = (1-\alpha)e^t + \alpha t^{1-\alpha} e^t$

(6) ${}_0^C\tilde{T}_\alpha(\sin t) = {}_0^{RL}\tilde{T}_\alpha(\sin t) = (1-\alpha)\sin t + \alpha t^{1-\alpha}\cos t$

(7) ${}_0^C\tilde{T}_\alpha(\cos t) = (1-\alpha)(\cos t - 1) - \alpha t^{1-\alpha}\sin t$

(8) ${}_0^{RL}\tilde{T}_\alpha(\cos t) = (1-\alpha)\cos t - \alpha t^{1-\alpha}\sin t$

(9) ${}_0^{RL}\tilde{T}_\alpha\left(e^{\left(\frac{1}{\alpha}t^\alpha\right)}\right) = e^{\left(\frac{1}{\alpha}t^\alpha\right)}$.

Results and Discussions

Instead of using fractional integrals, the improved conformable fractional derivative employs limit in the definition as in the conformable fractional derivative. Khalil et al. [14] provide the relationship between α-differentiable functions (in the sense of conformable derivative) and continuous functions. The improved conformable fractional derivative inherits this property as stated in the following theorems.

Theorem 7. Let $0 \le \alpha \le 1$. If f is an α-differentiable function in the sense of the improved Caputo-type conformable fractional derivative at t_0, then f is continuous at t_0.

Proof. By virtue of Definition 3, we have

$$\left({}_a^C\tilde{T}_\alpha f\right)(t_0) = (1-\alpha)\left(f(t_0) - f(a)\right) + \alpha \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon(t_0 - a)^{1-\alpha}) - f(t_0)}{\varepsilon}.$$

If we let $h = \varepsilon(t_0 - a)^{1-\alpha}$, then we get

$$\begin{aligned} \lim_{h \to 0} f(t_0 + h) - f(t_0) &= \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon(t_0 - a)^{1-\alpha}) - f(t_0)}{\varepsilon} . \varepsilon \\ &= \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon(t_0 - a)^{1-\alpha}) - f(t_0)}{\varepsilon} . \lim_{\varepsilon \to 0} \varepsilon \end{aligned}$$

$$= \frac{1}{\alpha}\left[\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(t_0) + (\alpha-1)\big(f(t_0)-f(a)\big)\right].0$$
$$= 0.$$

This implies that $\lim_{h\to 0} f(t_0+h) = f(t_0)$. Therefore, f is continuous at t_0.

Using analogous way in the proof of Theorem 7, we can easily obtain the continuity property of the improved Riemann-Liouville-type conformable fractional derivative as stated in the Theorem 8.

Theorem 8. Let $0 \le \alpha \le 1$. If f is an α-differentiable function in the sense of the improved Riemann-Liouville-type conformable fractional derivative at t_0, then f is continuous at t_0.

Proof. Using Definition 4 and letting $h = \varepsilon(t_0-a)^{1-\alpha}$, we have

$$\lim_{h\to 0} f(t_0+h) - f(t_0) = \frac{\varepsilon}{\alpha}\left[\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(t_0) + (\alpha-1)f(t_0)\right].$$

Letting ε goes to 0 will lead to $\lim_{h\to 0} f(t_0+h) = f(t_0)$, and hence it is continuous at t_0.

In the sense of classical derivative there exists a relation between an extremum point of a function and its derivative at the point. In this paper, we provide similar property for the improved conformable fractional derivative.

Theorem 9. Let $a > 0$ and c be an interior point of the interval $[a,b]$ at which $f:[a,b]\to\mathbb{R}$ has a relative extremum. If α-derivative of f at c exists, then $\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(c) = (1-\alpha)\big(f(c)-f(a)\big)$ and $\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(c) = (1-\alpha)f(c)$.

Proof. Assume that f has a relative maximum at c. The case of $f(c)$ is a relative minimum value is proven in similar way. Suppose that $\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(c)$ exists, then

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(c) = (1-\alpha)\big(f(c)-f(a)\big) + \alpha\lim_{\varepsilon\to 0}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon}.$$

Since $f(c)$ is a locally maximum value then

$$\lim_{\varepsilon\to 0^+}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon} \le 0,$$

and

$$\lim_{\varepsilon\to 0^-}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon} \ge 0.$$

This gives

$$\lim_{\varepsilon\to 0}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0^-}\frac{f(c+\varepsilon(c-a)^{1-\alpha}) - f(c)}{\varepsilon}$$
$$= 0.$$

Hence, we have

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(c) = (1-\alpha)\big(f(c)-f(a)\big).$$

Similarly, suppose that $({}_{a}^{RL}\tilde{T}_{\alpha}f)(c)$ exists. Using the fact that $\lim_{\varepsilon\to 0}\frac{f(c+\varepsilon(c-a)^{1-\alpha})-f(c)}{\varepsilon}$ must be 0, then

$$\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(c)=(1-\alpha)f(c)+\alpha\lim_{\varepsilon\to 0}\frac{f(c+\varepsilon(c-a)^{1-\alpha})-f(c)}{\varepsilon}=(1-\alpha)f(c).$$

It completes the proof.

If f is 1-differentiable (that is differentiable in the sense of classical derivative), then we can see from the relation of α-derivative and derivative of a function that

$$\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(t)=(1-\alpha)\big(f(t)-f(a)\big)+\alpha(t-a)^{1-\alpha}f'(t),$$

and

$$\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(t)=(1-\alpha)f(t)+\alpha(t-a)^{1-\alpha}f'(t),$$

we have $f'(c)=0$. This is in line with the result for the classical derivative. Now we provide some examples to elaborate further on the effectiveness of Theorem 9.

Example 10. Let $0\le\alpha\le 1$.

1. Suppose that $f(t)=\sin t$. We will evaluate $\left({}_{0}^{C}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)$ and $\left({}_{0}^{RL}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)$. Since $t=\frac{\pi}{2}$ is a point of relative maximum, then based on Theorem 9 we have

$$\left({}_{0}^{C}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)=(1-\alpha)\left(\sin\frac{\pi}{2}-\sin 0\right)=1-\alpha$$

$$\left({}_{0}^{RL}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)=(1-\alpha)\sin\frac{\pi}{2}=1-\alpha.$$

   If we calculate directly from the definition or from Theorem 6, we get

$$\left({}_{0}^{C}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)=\left({}_{0}^{RL}\tilde{T}_{\alpha}f\right)\left(\frac{\pi}{2}\right)=(1-\alpha)\sin\frac{\pi}{2}+\alpha\left(\frac{\pi}{2}\right)^{1-\alpha}\cos\frac{\pi}{2}=1-\alpha.$$

2. Suppose that $g(t)=\cos t$. We will evaluate $\left({}_{0}^{C}\tilde{T}_{\alpha}g\right)(\pi)$ and $\left({}_{0}^{RL}\tilde{T}_{\alpha}g\right)(\pi)$. Since $t=\pi$ is a point of relative minimum, then based on Theorem 9 we have

$$\left({}_{0}^{C}\tilde{T}_{\alpha}g\right)(\pi)=(1-\alpha)(\cos\pi-\cos 0)=-2(1-\alpha),$$

$$\left({}_{0}^{RL}\tilde{T}_{\alpha}g\right)(\pi)=(1-\alpha)(\cos\pi)=-(1-\alpha).$$

   Meanwhile, if we compute using Theorem 6, we obtain

$$\left({}_{0}^{C}\tilde{T}_{\alpha}g\right)(\pi)=(1-\alpha)(\cos\pi-\cos 0)-\alpha\pi^{1-\alpha}\sin\pi=-2(1-\alpha),$$

   and

$$\left({}_{0}^{RL}\tilde{T}_{\alpha}g\right)(\pi)=(1-\alpha)\cos\pi-\alpha\pi^{1-\alpha}\sin\pi=-(1-\alpha).$$

Khalil et al. [14] provide Rolle's theorem for the conformable derivative, while Riemann-Liouville and Caputo fractional derivative do not satisfy Rolle's theorem. In the following theorem, we present a property close to Rolle's theorem for the improved conformable fractional derivative.

Theorem 11. Let $a>0$ and $f:[a,b]\to\mathbb{R}$ be a function which satisfies

(1) f is continuous on $[a,b]$

(2) f is α-differentiable on (a,b) for some $\alpha\in(0,1)$

(3) $f(a)=f(b)$.

Then there exists $c\in(a,b)$ such that $\left({}_{a}^{C}\tilde{T}_{\alpha}f\right)(c)=(1-\alpha)\big(f(c)-f(a)\big)$ and $\left({}_{a}^{RL}\tilde{T}_{\alpha}f\right)(c)=(1-\alpha)f(c)$.

Proof. Suppose that f is a constant function, that is $f(t) = f(a)$. This guarantees that there exists at least one point in (a, b), called c, such that

$$\left({}_a^C\tilde{T}_\alpha f\right)(c) = {}_a^C\tilde{T}_\alpha\big(f(a)\big) = 0 = (1-\alpha)\big(f(c) - f(a)\big),$$

and

$$\left({}_a^{RL}\tilde{T}_\alpha f\right)(c) = {}_a^{RL}\tilde{T}_\alpha\big(f(a)\big) = (1-\alpha)f(a) = (1-\alpha)f(c).$$

Now suppose that f is not a constant function. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, then f has a relative extremum in (a, b). Let $c \in (a, b)$ be a point of relative extremum. By virtue of Theorem 9, we get $\left({}_a^C\tilde{T}_\alpha f\right)(c) = (1-\alpha)\big(f(c) - f(a)\big)$ and $\left({}_a^{RL}\tilde{T}_\alpha f\right)(c) = (1-\alpha)f(c)$.

From Theorem 11, we can see that an α-differentiable function in the sense of the improved Caputo-type conformable fractional derivative satisfies Rolle's theorem as in the classical derivative if the function is constant. However, it is slightly different for the improved Riemann-Liouville-type conformable fractional derivative. An α-differential function in the sense of the improved Riemann-Liouville-type conformable fractional derivative does not satisfy the Rolle's theorem as for the classical derivative only for identically vanishing function $f(t) = 0$ for all $t \in [a, b]$. Surprisingly, if f is 1-differentiable then we can obtain $f'(c) = 0$ for both the improved conformable fractional derivatives. This result is in line with the classical derivative.

The conformable derivative has been proven to satisfy the mean value theorem. However, this is not the case for Riemann-Liouville and Caputo fractional derivatives. Both commonly used fractional derivatives do not have mean value theorem. In this paper, we give a theorem close to the mean value theorem for the improved Riemann-Liouville-type fractional derivative.

Theorem 12. Let $a > 0$ and $f: [a, b] \to \mathbb{R}$ be a function that satisfies

(1) f is continuous on $[a, b]$

(2) f is α-differentiable on (a, b) for some $\alpha \in (a, b)$ in the sense of the improved Riemann-Liouville-type fractional derivative.

Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = \left[\left({}_0^{RL}\tilde{T}_\alpha f\right)(c) + (\alpha - 1)f(c)\right]\left(\frac{1}{\alpha}e^{\frac{1}{\alpha}(b^\alpha - c^\alpha)} - \frac{1}{\alpha}e^{\frac{1}{\alpha}(a^\alpha - c^\alpha)}\right).$$

Proof. Consider the function $g(t)$ defined on $[a, b]$ as the following.

$$g(t) = f(t) - f(a) - \frac{f(b) - f(a)}{e^{\frac{1}{\alpha}b^\alpha} - e^{\frac{1}{\alpha}a^\alpha}}\left(e^{\frac{1}{\alpha}t^\alpha} - e^{\frac{1}{\alpha}a^\alpha}\right).$$

Note that g is continuous on $[a, b]$, α-differentiable on (a, b), and $g(a) = g(b)$. Based on the Theorem 11, there exists $c \in (a, b)$ such that

$$\begin{aligned}\left({}_0^{RL}\tilde{T}_\alpha g\right)(c) &= (1-\alpha)g(c)\\ &= (1-\alpha)\left[f(c) - f(a) - \frac{f(b) - f(a)}{e^{\frac{1}{\alpha}b^\alpha} - e^{\frac{1}{\alpha}a^\alpha}}\left(e^{\frac{1}{\alpha}c^\alpha} - e^{\frac{1}{\alpha}a^\alpha}\right)\right].\end{aligned}$$

On the other hand,

$$\left({}_0^{RL}\tilde{T}_\alpha g\right)(c) = \left({}_0^{RL}\tilde{T}_\alpha f\right)(c) - (1-\alpha)f(a) - \frac{f(b) - f(a)}{e^{\frac{1}{\alpha}b^\alpha} - e^{\frac{1}{\alpha}a^\alpha}}\left(e^{\frac{1}{\alpha}c^\alpha} - (1-\alpha)e^{\frac{1}{\alpha}a^\alpha}\right).$$

From the two above equations, by simplifying we have

$$\left({}_0^{RL}\tilde{T}_\alpha f\right)(c) = (1-\alpha)f(c) + \alpha \frac{f(b)-f(a)}{e^{\frac{1}{\alpha}b^\alpha} - e^{\frac{1}{\alpha}a^\alpha}}\, e^{\frac{1}{\alpha}c^\alpha}$$

$$= (1-\alpha)f(c) + \frac{f(b)-f(a)}{\frac{1}{\alpha}e^{\frac{1}{\alpha}(b^\alpha - c^\alpha)} - \frac{1}{\alpha}e^{\frac{1}{\alpha}(a^\alpha - c^\alpha)}}.$$

Thus, we get

$$f(b) - f(a) = \left[\left({}_0^{RL}\tilde{T}_\alpha f\right)(c) + (\alpha - 1)f(c)\right]\left(\frac{1}{\alpha}e^{\frac{1}{\alpha}(b^\alpha - c^\alpha)} - \frac{1}{\alpha}e^{\frac{1}{\alpha}(a^\alpha - c^\alpha)}\right).$$

It completes the proof.

Theorem 12 can be used to characterize a constant function based on its α-derivative in the sense of the improved Riemann-Liouville-type conformable fractional derivative. The theorem is given in the following.

Theorem 13. Let $a > 0$ and $f: [a,b] \to \mathbb{R}$ be a function which satisfies

(1) f is continuous on $[a,b]$

(2) f is α-differentiable on (a,b) for some $\alpha \in (0,1)$

(3) $\left({}_0^{RL}\tilde{T}_\alpha f\right)(t) = (1-\alpha)f(t)$ for all $t \in (a,b)$.

Then f is constant on $[a,b]$.

Proof. We will show that $f(t) = f(a)$ for all $t \in [a,b]$. For $t > a$, by virtue of Theorem 12, there exists $c \in (a,b)$ which is depending on t such that

$$f(t) - f(a) = \left[\left({}_0^{RL}\tilde{T}_\alpha f\right)(c) + (\alpha - 1)f(c)\right]\left(\frac{1}{\alpha}e^{\frac{1}{\alpha}(t^\alpha - c^\alpha)} - \frac{1}{\alpha}e^{\frac{1}{\alpha}(a^\alpha - c^\alpha)}\right)$$

$$= \left[(1-\alpha)f(c) + (\alpha - 1)f(c)\right]\left(\frac{1}{\alpha}e^{\frac{1}{\alpha}(t^\alpha - c^\alpha)} - \frac{1}{\alpha}e^{\frac{1}{\alpha}(a^\alpha - c^\alpha)}\right)$$

$$= 0.$$

Hence, we have $f(t) = f(a)$ for any $t \in [a,b]$. Therefore f is a constant function.

In the classical derivative, if there are two continuous functions whose derivatives are the same in a closed interval, then the functions have a constant difference. This property is a result of the derivative of a constant function. However, this result does not apply to the improved Riemann-Liouville-type conformable fractional derivative. From Theorem 13, we can deduce that two α-derivatives of α-differentiable functions have a constant difference if and only if the two functions have also a constant difference.

Conclusion

The relation between α-differentiable functions and its continuity has been provided. In terms of continuity property, the improved conformable fractional derivative and the conformable derivative have the same property as the classical derivative. The improved conformable fractional derivative as a modification of the conformable derivative has corresponding Rolle's theorem and mean value theorem. We can conclude that alongside be a better approximation to Riemann-Liouville and Caputo fractional derivative, the improved conformable fractional derivative has nicer properties regarding Rolle's theorem and mean value theorem compared to Riemann-Liouville and Caputo fractional derivatives.

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